NOTE ON PRODUCTS OF CONSECUTIVE INTEGERS

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It has been conjectured that the product

$$A_k(n) = n(n+1) \dots (n+k-1)$$

of k consecutive positive integers is never an l-th power, if $k > 1$ and $l > 1$. This is well known for $k = 2$ and $k = 3$, and was recently proved by G. Szekeres† for $k \leqslant 9$. It has also been proved by Narumi‡ for $l = 2$ and $k \leqslant 202$. In this note we prove the conjecture for $l = 2$ and all k; that is, we prove that *a product of consecutive integers is never a square*. The method is similar to that used by Narumi.

Suppose that

$$(1) \qquad A_k(n) = n(n+1) \dots (n+k-1) = y^2.$$

Then, clearly,

$$n+i = a_i x_i^2 \quad (i = 0, 1, \dots, k-1),$$

where the a's are quadratfrei integers whose prime factors are all less than k (since a prime not less than k must divide $n+i$ to an even power). The idea of the proof consists in showing that the numbers a_i are all different, and in deducing from this a contradiction.

As a preliminary, we show§ that $n > k^2$. Suppose first that $n \leqslant k$. Then, by a theorem of Tchebycheff, there exists a prime p satisfying $n+k > p \geqslant \frac{1}{2}(n+k) \geqslant n$, and from this it follows that $p \mid A_k(n)$, $p^2 \nmid A_k(n)$, which is impossible. Suppose now that $n > k$. Then, by a theorem of Sylvester and Schur‖, $A_k(n)$ has a prime factor $q > k$. Thus, for some i, $q^2 \mid n+i$; whence

$$n+i \geqslant (k+1)^2, \quad n > k^2.$$

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* Received 7 February, 1939; read 16 February 1939.

† Oral communication.

‡ Seimatsu Narumi, *Tôhoku Math. Journal*, 11 (1917), 128–142.

§ See R. Obláth, *Tôhoku Math. Journal*, 38 (1933), 73–92.

‖ P. Erdös, *Journal London Math. Soc.*, 9 (1934), 282–288.

Suppose that the a's are not all different, say that $a_i = a_j$, where, without loss of generality, $i > j$. Then

$$\begin{aligned} k > a_i x_i^2 - a_i x_j^2 = a_i(x_i^2 - x_j^2) &> 2a_i x_j \\ &\geqslant 2\sqrt{(a_i x_j^2)} \\ &= 2\sqrt{(n+j)} \\ &> \sqrt{n}, \end{aligned}$$

which we have proved to be impossible. Hence the a's are all different.

It follows that the product of the a's is greater than or equal to the product of the first k quadratfrei numbers. For $m \geqslant 9$, the number of quadratfrei numbers not exceeding m is at most

$$m - [\tfrac{1}{4}m] - 1 < \tfrac{3}{4}m.$$

Hence, for $r \geqslant 7$, the r-th quadratfrei number is greater than $4r/3$. Now* the product of the first 24 quadratfrei numbers is greater than $(\frac{4}{3})^{24}\, 24!$. It follows by induction that, for $k \geqslant 24$, the product of the first k quadratfrei numbers is greater than $(\frac{4}{3})^k k!$ Hence

$$a_0 a_1 \ldots a_{k-1} > (\tfrac{4}{3})^k k!. \tag{2}$$

On the other hand, the number of a's divisible by a prime $p < k$ does not exceed $[k/p]+1$, and the a's are quadratfrei. Hence the power to which p divides $a_0 a_1 \ldots a_{k-1}$ does not exceed $[k/p]+1$. Further, if p lies in one of the intervals

$$\frac{k}{2l} \geqslant p > \frac{k}{2l+1} \quad (l = 1,\ 2,\ \ldots),$$

the number $[k/p]+1 = 2l+1$ is odd, whereas the power to which p divides $a_0 a_1 \ldots a_{k-1}$ is even, since this is a square. Hence the power to which such a prime divides $a_0 a_1 \ldots a_{k-1}$ does not exceed $[k/p]$, and this conclusion is

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* It is sufficient to prove that

$$\frac{26.29.30.31.33.34.35.37}{4.8.9.12.16.18.20.24} > (\tfrac{4}{3})^{24}.$$

Now the left-hand side can be written as

$$(\tfrac{78}{29})(\tfrac{29}{12})(\tfrac{29}{12})(\tfrac{31}{12})(\tfrac{33}{12})(\tfrac{34}{12})(\tfrac{37}{12})(\tfrac{39}{16}),$$

and here every factor is greater than $(\frac{4}{3})^3$.

easily seen to hold also in the case $p = k/(2l+1)$. Hence we have

$$(3) \qquad a_0 a_1 \dots a_{k-1} \leqslant \prod_{p<k} p^{[k/p]} \prod_{k>p>\frac{1}{2}k} p \prod_{\frac{1}{3}k>p>\frac{1}{4}k} p \dots .$$

We now prove that

$$(4) \qquad \prod_{k>p>\frac{1}{2}k} p \prod_{\frac{1}{3}k>p>\frac{1}{4}k} p \dots \quad \text{divides} \quad \binom{k-1}{[\frac{1}{2}(k-1)]}.$$

Let $u = [\frac{1}{2}(k-1)]$, $v = k-1-u$. It is well known that the exact power to which p divides the above binomial coefficient is

$$\sum_{\nu=1}^{\infty} \left\{ \left[\frac{k-1}{p^\nu}\right] - \left[\frac{u}{p^\nu}\right] - \left[\frac{v}{p^\nu}\right] \right\}.$$

Each term in this series is non-negative, hence it is sufficient to prove that, if

$$\frac{k}{2l-1} > p > \frac{k}{2l},$$

then

$$\left[\frac{k-1}{p}\right] > \left[\frac{u}{p}\right] + \left[\frac{v}{p}\right].$$

Obviously, $[(k-1)/p] = 2l-1$. Hence it is sufficient to prove that

$$\left[\frac{u}{p}\right] = \left[\frac{v}{p}\right].$$

If k is odd, we have $u = v$. If k is even, we have $v = u+1 = \frac{1}{2}k$, and, since $p \nmid k$, we have $p \nmid u+1$. This proves (4). By (3) and (4),

$$(5) \qquad a_0 a_1 \dots a_{k-1} \leqslant \binom{k-1}{[\frac{1}{2}(k-1)]} \prod_{p<k} p^{[k/p]}$$

$$\leqslant 2^{k-2} \prod_{p \leqslant k} p^{[k/p]}.$$

By a well-known theorem of Legendre, if

$$k = c_0 p^s + c_1 p^{s-1} + \dots + c_s \quad (0 \leqslant c_i \leqslant p-1),$$

the exact power to which p divides $k!$ is

$$r_p = \frac{k - \Sigma c_i}{p-1} \geqslant \frac{k}{p-1} - (s+1).$$

Thus
$$p^{r_p} \geqslant \frac{p^{k/p-1}}{p^{s+1}} \geqslant \frac{p^{k/p-1}}{kp}.$$

Using this result for $p=2$, 3, and the trivial result $r_p \geqslant [k/p]$ for $p>3$, we obtain

$$k! \geqslant \frac{2^k}{2k}\,\frac{3^{\frac{1}{2}k}}{3k} \prod_{3<p\leqslant k} p^{[k/p]}. \tag{6}$$

By (2), (5), (6),
$$2^{k-2}\,2^{[k/2]}\,3^{[k/3]} > (\tfrac{4}{3})^k \frac{2^k\,3^{\frac{1}{2}k}}{6k^2},$$

whence
$$2^{k-2} > (\tfrac{4}{3})^k \frac{2^{\frac{1}{2}k}\,3^{\frac{1}{6}k}}{6k^2},$$

i.e.

$$(\tfrac{3}{2})^6\,k^{12}\,3^{5k} > 2^{9k}. \tag{7}$$

Since $2^8 > 3^5$, (7) does not hold if
$$2^k > (\tfrac{3}{2})^6\,k^{12},$$
and this is the case* for $k \geqslant 100$. The remaining cases ($k<100$) can easily be settled by special arguments; in fact, as already stated, Narumi settled all cases for which $k \leqslant 202$.

By similar arguments, involving slightly longer calculations, we can prove the following theorem. Take $k>3$, and let
$$A_i \quad (i=0,\,1,\,\ldots,\,k-1)$$
be the product of all the powers of primes less than k composing $n+i$. Let $A_i = a_i x_i^2$, where a_i is quadratfrei. Then a_0, a_1, ..., a_{k-1} cannot be all different. This result is more general than that proved above, since we do not suppose that $a_0 a_1 \ldots a_{k-1}$ is a square. From this result it immediately follows that, for $k>3$ and $n \geqslant k$, at least one of the integers $n, n+1, \ldots, n+k-1$ is divisible by a prime $p>k$ with an odd exponent.

By similar arguments, we can prove that a product of consecutive *odd* integers is never a power.

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* We have $2^{100} = (2^{10})^{10} > (1000)^{10} = 10^{30}$, $(\frac{3}{2})^6 (100)^{12} < 100.10^{24} = 10^{26}$.

Also, if we replace k by $k+1$, the left-hand side of the inequality is multiplied by 2, and the right-hand side by
$$\left(1+\frac{1}{k}\right)^{12} \leqslant \left(1+\frac{1}{100}\right)^{12} < 2.$$

[*Added* 6 *May*, 1939. Since writing this paper I have learned that O. Rigge has proved the following more general result. Let $n > 1$ be an integer. Then, if all prime factors of c are not greater than $\frac{1}{2}n$, the equation

$$c(x+1)(x+2)\dots(x+n) = y^2$$

has no solutions. Rigge's proof is similar to mine.]

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